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MORE CONSISTENCY RESULTS IN PARTITION CALCULUS*

BY

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ABSTRACT

This paper has two principal aims. The first is to supply a proof of Theorem 6 of [ShSt1]:

THEOREM: *If ZFC+ "there are c^+ measurable cardinals" is consistent, then so is ZFC+ " $\aleph_{c^+}$ is not a strong limit cardinal and $\aleph_{c^+} \rightarrow (\aleph_{c^+}, \aleph_1)^2$ ".*

This is done in sections 1 and 2. See the introduction for a discussion of the evolution of the proof and of some interesting questions which remain open, related to obstacles encountered in obtaining maximum freedom in arranging for any desired cardinal exponentiation in Theorems 4 and 6 of [ShSt1]. The method is quite generally applicable in partition calculus and variants of it have in fact been applied in recent work of the authors, see [ShSt2]. First, a preservation result is proved for the game-theoretic properties of the filters considered in [ShSt1]. Then, it is shown that the existence of a system of such filters yields a canonization-style result. Finally, it is shown that the canonization property gives the positive partition relation. The second aim makes the title of this paper

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slightly inaccurate (but we suspect this will be pardoned): we supply a (straightforward) proof of a result which shows that Theorem 2 of [ShSt1] in some sense is best possible. This is done in section 3.

In section 1, we prove general preservation theorems, (1.4), (1.8), for the game-theoretic properties of filters considered in section 5 of [ShSt1]. The argument in (5.6) of [ShSt1] is a prototypical special case for the "local" version of the property corresponding to weakly compact cardinals. At the time [ShSt1] was written we thought that we could prove a stronger version of these preservation theorems than the ones we actually give in section 1; we thought it was possible to omit the hypothesis that the forcing conditions have the appropriate degree of strategic-closure. In February, 1988, J.-P. Levinski pointed out to the second author the mistake in our proof, and furnished an argument, which we present with his permission, in (1.9), showing that some such hypothesis, at least about the appropriate degree of the Baire property of the forcing conditions, is indeed necessary.

We had intended to apply the stronger forms of the preservation results to partial orderings which arranged for various patterns of cardinal exponentiation as well as collapsing the large cardinals, thereby planning to obtain models of Theorems 4 and 6 of [ShSt1] having any desired pattern of cardinal exponentiation consistent with some limitative results. Of course, lacking the stronger forms of the preservation theorems, we cannot produce all of the models we hoped for. However, in quite recent work, [ShSt2], we have shown that, assuming GCH in the ground model, by adding $\aleph_{\aleph_2}$ Cohen subsets of $\aleph_1$, we obtain a model where CH + $2^{\aleph_1} = \aleph_{\aleph_2}$ + $\aleph_{\aleph_2} \to (\aleph_{\aleph_2}, \aleph_1)^2$. Of course, this does not supersede the results of this paper, since, in this paper, we obtain the consistency of the positive relation with a different pattern of cardinal exponentiation, in addition to obtaining some actual implications, namely that the existence of the appropriate system of filters implies the canonization property which, in turn, implies that $\aleph_{c^+} \to (\aleph_{c^+}, \aleph_1)^2$. However, we do not currently know how, even starting from a measurable cardinal, for example, to produce a model where $2^{\aleph_1} > \aleph_2$ and $\omega_3\omega_1 \to (\omega_3\omega_1, 3)^2$.

In section 2, we have cardinal parameters θ, δ, λ, where $2^{<\delta} < \theta = \mathrm{cf}\ \lambda < \lambda$, $\theta \to (\theta, \delta)^2$, and we assume that we can choose a monotone sequence $\langle \lambda_\alpha : \alpha < \theta \rangle$ of regular cardinals cofinal in λ, with $\lambda_0 > \theta$, such that for each α, $\lambda_\alpha \to (\lambda_\alpha, \delta)^2$.

For Theorem 6 of [ShStl], of course $\delta = \aleph_1$, $\theta = c^+$, $\lambda = \aleph_{c^+}$ and we can take the λ_α to simply be the regular alephs between θ and λ (however, we should note, see below, that in the models we produce, not only will $\lambda_0 > \theta$, in fact $\lambda_0 > 2^\theta$; this is imposed by the limitations of the preservation theorem, not required for the arguments of section 2). We will have $\lambda_\alpha \to (\lambda_\alpha, \delta)^2$ since we'll have $\delta =$ the base λ_α logarithm of λ_α, viz. [EHMR], section 17. We introduce a canonization property, (CP), and show, in (2.1), that in the presence of our other hypotheses it implies that $\lambda \to (\lambda, \delta)^2$. The remainder of section 2 is devoted to proving, in (2.3), that (CP) is a consequence of the existence of filters, $\mathcal{F}_\alpha$ on λ_α which are λ_α-complete and such that the partial ordering of the positive sets with reverse inclusion (recall our convention about the direction of the partial ordering!) is $\theta + 1$-strategically-closed.

Of course, in our application to Theorem 6 of [ShSt1], our model is the generic extension obtained by collapsing c^+ measurable cardinals to be the λ_α, while simultaneously making λ $(= \aleph_{c^+}) \leq 2^\kappa$, for some $c^+ < \kappa < \lambda$. Arranging for this cardinal exponentiation is not necessary for the arguments of section 2, but is required for our result to have any interest since, as already mentioned in the third paragraph, p. 125 of [ShSt1], it was already known, [EHMR], that $\aleph_{c^+} \to (\aleph_{c^+}, \aleph_1)^2$, if $\aleph_{c^+}$ is a strong limit cardinal. The restriction that $c^+ < \kappa$ is imposed by our inability to preserve the required game theoretic property of the filters while adding sequences of length $\leq c^+$, see the above discussion, and (1.9), below. In fact, we must have $2^{(c^+)} < \lambda_0$; there are no other restrictions on λ_0 except that it must be a regular uncountable cardinal $< \aleph_{c^+}$. The $\mathcal{F}_\alpha$ are the filters generated in the generic extension by pre-specified normal measures on the formerly measurable cardinals. The preservation theorem, (1.4), guarantees the $\theta + 1$ strategic closure of the partial ordering of positive sets, with reverse inclusion. For a filter $\mathcal{F}$, this ordering will be denoted by $\mathbb{P}(\mathcal{F})$ (or simply $\mathbb{P}$, of course, when no confusion lurks).

The material of section 3 is elementary, as we have already said. We should take this opportunity to correct some inaccuracies in the historical remarks of [ShSt1]. In the second paragraph of p. 125, the correct reference to the first author's work on bounds for $2^{\aleph_0}$ is [PropFor], Chapter XIII, not reference 9 of [ShSt1]; also Section 47 of [EHMR] presents older improvements, including those of reference 9 of [ShSt1], by the first author, of results of Galvin–Hajnal, et al., but **not** the material of [PropFor], Chapter XIII.

All but one of the problems listed as open in [ShSt1] alas remain so, despite partial progress on some of them. In July, 1988, the second author and D. Velleman, elaborating on work of Miyamoto in his dissertation [My], succeeded in proving that if, e.g., $2^{\aleph_1} = \aleph_2$ and $\omega_3\omega_1 \to (\omega_3\omega_1, 3)^2$, then either $\aleph_2$ or $\aleph_3$ is $(\text{inaccessible})^L$. A similar result was proved independently by C. Morgan.

This involves showing that if there's an $(\aleph_2, 1)$ morass with linear limits and a complete amalgamation system (see [V]), then $\omega_3\omega_1 \to (\omega_3\omega_1, 3)^2$. However, with mild hypotheses on A, Donder's construction in L of a $(\kappa, 1)$ morass with linear limits, for regular non-weakly compact uncountable κ (see [D]) can in fact be carried out in $L[A]$, where $A \subseteq \kappa$. An A satisfying these hypotheses can be found, if, for example, κ is a $(\text{successor cardinal})^L$ (much weaker conditions suffice to give us an A as required). For $\kappa = \aleph_2$, such an A can also, as usual, be chosen to make $\aleph_3$ the real $\aleph_3$, if the real $\aleph_3$ is not $(\text{inaccessible})^L$. Thus, if neither $\aleph_2$ nor $\aleph_3$ is $(\text{inaccessible})^L$, we can find a suitable A and an $(\omega_2, 1)$ morass constructed in $L[A]$ for such an A, is, in V, an $(\omega_2, 1)$ morass with linear limits. By our hypotheses on cardinal exponentiation, there is also, in V, a complete amalgamation system (again, see [V]). The conclusion is then clear. This work appears as [SVM].

1. Preservation and anti-preservation theorems

Our first order of business is to prove results which say that filter existence properties of the type considered in section 5 of [ShSt1] are preserved by certain well-behaved forcing extensions. We first recall the property $\Pr(\lambda, \mathcal{F}, \kappa)$.

(1.1) Let $\omega \leq \kappa$ (in [ShSt1] we assumed that κ was uncountable, which corresponded to the intended applications, but is not necessary), $\kappa^+ < \lambda$, λ regular, and let $\alpha \leq \kappa^+$ (in the simplest setting, $\lambda = \kappa^{++}$. Let $\mathcal{F}$ be a filter on λ. Let $A_0 = \lambda$. $G(\lambda, \mathcal{F}, \alpha)$ is the game: EMPTY and NONEMPTY pick $\mathcal{F}$-positive sets A_ξ, $0 < \xi < \delta \leq \alpha$, generating a $\subseteq$-decreasing sequence, where EMPTY plays at odd stages, NONEMPTY at non-zero even stages, and NONEMPTY loses if for some non-zero even $\delta < \alpha$, NONEMPTY has no legal move (in which case δ must be a positive limit ordinal). $\Pr(\lambda, \mathcal{F}, \alpha)$ is the statement: $\mathcal{F}$ is a normal filter on λ with the property that NONEMPTY has a winning strategy in $G(\lambda, \mathcal{F}, \alpha)$. In fact, $\Pr(\lambda, \mathcal{F}, \alpha)$ is just the statement that ($\mathcal{F}$ is normal and) $\mathbb{P}(\mathcal{F})$ is α-strategically-closed. We shall use this formulation here and below. We shall be interested in the weaker statement that results when the requirement of normality is weakened to mere uniformity. $\Pr'(\lambda, \kappa)$ is the statement that there's a uniform filter $\mathcal{F}$ on λ for which $\Pr'(\lambda, \mathcal{F}, \kappa + 1)$ holds, i.e. $\mathbb{P}(\mathcal{F})$ is $\kappa + 1$-strategically-closed. We first define the abstract version of the $\vec{p}$ in I of (5.6) of [ShSt1].

(1.2) Definition: Let $\mathbb{P}$ be a partial ordering, $\mathcal{F}$ a filter on λ, let S^1 be $\mathcal{F}$-positive, let $\bar{p} \in P$ and let $\vec{p} = (p_\alpha : \alpha \in S^1)$, where each $p_\alpha \in P$. $\vec{p}$ is **$\bar{p}$-orderly** if $\bar{p} \leq p_\alpha$, for all $\alpha \in S^1$, and for all $q \in P$, q is compatible with $\bar{p}$ iff q is compatible with p_α for $\mathcal{F}$-almost-all $\alpha \in S_1$. ∎

Notice that when $\vec{p}$ is $\bar{p}$-orderly each p_α, $\alpha \in S^1$, is compatible with $\mathcal{F}$-almost all of the p_β, $\beta \in S^1$, as can be easily seen, taking $q = p_\alpha$. Thus $\mathcal{F}$-c.c, defined below, strongly implies λ-c.c.

(1.3) Definition: Let $\mathbb{P}$, $\mathcal{F}$ be as in (1.2). $\mathbb{P}$ is $\mathcal{F}$-c.c iff whenever $S \in \mathcal{F}^+$, $\vec{p} \in {}^S P$ there's $\mathcal{F}$-positive $S^1 \subseteq S$ and a **largest** $\bar{p} \in P$ such that $\bar{p} \leq p_\alpha$, for all $\alpha \in S^1$, and for this $\bar{p}$, $\vec{p}|S^1$ is $\bar{p}$-orderly. ∎

Notice that the proof of Fact a of (5.6) of [ShSt1] is essentially a proof that the Lévy collapse is E-c.c. (essentially, because in (5.6) of [ShSt1], we were working with the "local version" $Q(\lambda, \kappa)$ of $P(\lambda, \kappa)$), but see below, (1.5)–(1.8), where we rework this proof. In practice, of course, the $\mathcal{F}$-c.c. is usually established using the normality of $\mathcal{F}$.

(1.4) We now prove the basic preservation result.

THEOREM: *If $\mathbb{P}$ is $\mathcal{F}$-c.c. and $\alpha + 1$-strategically-closed, and $\mathbb{P}(\mathcal{F})$ is $\alpha + 1$-strategically-closed, where $\alpha < \lambda$ is a limit ordinal, then, in $V^{\mathbb{P}}$, $\mathbb{P}(\mathring{\mathcal{F}})$ is $\alpha + 1$-strategically-closed (here, of course, $\mathring{\mathcal{F}}$ is the canonical name for the filter generated by $\mathcal{F}$ in $V^{\mathbb{P}}$).*

Proof: This is essentially the proof in (5.6) of [ShSt1], starting from Fact b. As there, we introduce the sets $\mathring{S}(\vec{p})$, where $\vec{p} = (p^{\vec{p}}_\alpha : \alpha \in S) = (p_\alpha : \alpha \in S)$, with S $\mathcal{F}$-positive and each $p_\alpha \in P$: this is just the term "α if p_α", i.e. $\{\alpha \in S : p_\alpha \in \mathring{G}\}$. We first have the analogue of Fact c.

LEMMA A: *If $S^1 \in \mathcal{F}^+$, $\bar{p} \in P$ and $\vec{p} = (p_\alpha : \alpha \in S^1)$ is $\bar{p}$-orderly, $p \in P$ and p forces that $\mathring{S}$ is an $\mathring{\mathcal{F}}$-positive subset of $\mathring{S}(\bar{p})$, then there's $\mathcal{F}$-positive $S^2 \subseteq S^1$, $\bar{r} \in P$ and $\vec{r} = (r_\alpha : \alpha \in S^2)$ with $p \leq \bar{r}$, $p_\alpha \leq r_\alpha$, for $\alpha \in S^2$, such that p forces that $\mathring{S}(\vec{r}) \subseteq \mathring{S}$ and $\vec{r}$ is $\bar{r}$-orderly.*

Proof: Since p forces that $\mathring{S}$, and therefore $\mathring{S}(\vec{p})$, is $\mathring{\mathcal{F}}$-positive, clearly p is compatible with $\bar{p}$, and in fact this is true for all q with $p \leq q$. Thus, by the separativity of $\mathbb{P}$, $\bar{p} \leq p$.

$\mathring{S}$ can be represented as $(A_\alpha : \alpha < \lambda)$ where each A_α is an antichain. Since p forces that $\mathring{S} \subseteq \mathring{S}(\vec{p})$, we may assume, WLOG, that if $s \in A_\alpha$ and s is compatible with p, then $\alpha \in S^1$ and $p_\alpha \leq s$. Let $S^* = \{\alpha \in S^1 : (\exists s \in A_\alpha) s$ is compatible with $p\}$. Since p forces that $\mathring{S}$ is $\mathring{\mathcal{F}}$-positive, S^* is $\mathcal{F}$-positive. Let $(\mathring{S})^*$ be the name for the subset of λ which is represented by $(A^*_\alpha : \alpha \in S^*)$, where, for $\alpha \in S^*$, $A^*_\alpha = \{s \in A_\alpha : s$ is compatible with $p\}$. Thus, for $\alpha \in S^*$ and $s \in A^*_\alpha$, $p_\alpha \leq s$, and clearly p forces that $\mathring{S} = (\mathring{S})^*$. Further, for $\alpha \in S^*$ and $s \in A^*_\alpha$, replacing s by a maximal antichain in $\{q \in P : p, s \leq q\}$, we get A^{**}_α and the name $(\mathring{S})^{**}$ which is represented by $(A^{**}_\alpha : \alpha \in S^*)$, such that clearly p forces $(\mathring{S})^{**} = (\mathring{S})^*$. Thus, we may assume, WLOG, that for all $s \in A^*_\alpha$, $p \leq s$.

For $\alpha \in S^*$, choose $r_\alpha \in A^*_\alpha$; let $(\vec{r})^* = (r_\alpha : \alpha \in S^*)$. Since $\mathbb{P}$ is $\mathcal{F}$-c.c., let $S^2 \subseteq S^*$ be $\mathcal{F}$-positive and $\bar{r} \in P$ with $p \leq \bar{r}$ and $\vec{r} =_{\rm def} (\vec{r})^* | S^2$ which is $\bar{r}$-orderly. This is as required. ∎

We now complete the proof of the Theorem. Let σ, τ be winning strategies in V for NONEMPTY, respectively in $G(\lambda, \mathcal{F}, \alpha+1)$ and in the increasing sequence game of length $\alpha+1$ on $\mathbb{P}$. We describe a winning strategy in $V^{\mathbb{P}}$ for NONEMPTY in $G(\lambda, \mathring{\mathcal{F}}, \alpha + 1)$ based on σ and τ. The way the strategy works at limit stages is clearly analogous to the ideas of the proof of Fact b. from (5.6) of [ShSt1]; we have chosen to give an informal description of the strategy, free of the many indices a complete, formal description would have to carry along. We trust that we have given a clear enough account that the so-inclined reader will be able to supply the indices without difficulty.

NONEMPTY's strategy will involve playing sets of the form $\mathring{S}(\vec{r})$ while EMPTY will be playing names, $\mathring{S}$, of $\mathring{\mathcal{F}}$-positive sets and conditions, p, which force the names to be positive subsets of NONEMPTY's previous move. NONEMPTY will also play an auxiliary run of $G(\lambda, \mathcal{F}, \alpha + 1)$, using σ, against the domains of the $\vec{r}$ she has already played, viewed as EMPTY's moves in $G(\lambda, \mathcal{F}, \alpha + 1)$. This is so she will be able to find an $\mathcal{F}$-positive subset of the intersection of these domains when she has to move at a limit ordinal stage in the main game, $G(\lambda, \mathring{\mathcal{F}}, \alpha + 1)$. She will also play, using τ, λ auxiliary runs of the increasing sequence game on $\mathbb{P}$, one run for each $\beta < \lambda$. The lengths of these runs will vary; the βth run will stop as soon as β disappears from the $\mathcal{F}$-positive set NONEMPTY has just played at the current stage in $G(\lambda, \mathcal{F}, \alpha+1)$. If this is a successor stage, this can happen either because β has just disappeared from the domain of the last $\vec{r}$ which NONEMPTY has used on her previous move in $G(\lambda, \mathring{\mathcal{F}}, \alpha+1)$, or because it has just disappeared from her σ-move in $G(\lambda, \mathcal{F}, \alpha+1)$. At limit stages, β can disappear only for the second reason. The purpose of these runs is once again to handle limit stages; for $\beta \in$ the $\mathcal{F}$-positive set given by σ at this limit stage (see above), the candidate to be r_β will be obtained by playing according to τ in the βth run (this is only the candidate, because possibly β will disappear when this $\mathcal{F}$-positive set is thinned in applying $\mathcal{F}$-c.c., see below).

At successor stages, NONEMPTY will reply to $\mathring{S}$ and p using Lemma A, i.e., her plays at successor stages will be names of the form $\mathring{S}(\vec{r})$; at stage 2, she uses $(1_{\mathbb{P}} : \alpha < \lambda)$ as $\vec{p}$ in Lemma A. At limit stages $\delta \leq \alpha$, she first proceeds in the runs of the auxiliary games as outlined above. At this point, she has produced S, an $\mathcal{F}$-positive subset of the intersection of the domains of the *previous* $\vec{r}$, and the *current* $(r_\alpha : \alpha \in S)$. She uses the $\mathcal{F}$-c.c. of $\mathbb{P}$ to get $S^1 \in \mathcal{F}^+ \cap \mathcal{P}(S)$ and $\bar{p}$ such that $(r_\alpha : \alpha \in S^1)$ is $\bar{p}$-orderly. $\mathring{S}((r_\alpha : \alpha \in S^1))$ is her play at stage δ. Clearly this strategy is winning for $G(\lambda, \mathring{\mathcal{F}}, \alpha+1)$ in $V^{\mathbb{P}}$. This completes the proof of the Theorem. ∎

(1.5) We now turn to the preservation of the "local version", $Q(\lambda, \alpha)$ (which is actually argued for in (5.6) of [ShSt1]) of the principle $\exists \mathcal{F} \operatorname{Pr}(\lambda, \mathcal{F}, \alpha)$. We localize the notions and the proofs of (1.1)–(1.4), so we only indicate the main new features. It will be most convenient to deal directly with the $L_{\lambda\lambda}$ elementary substructures of the large models implicit in the statement of $Q(\lambda, \alpha)$ (and explicit in the proofs), rather than coding into the $\mathfrak{p} \subseteq \mathfrak{p}^* \subseteq \mathcal{P}(HF(\lambda))$ and the λ-many$<$ λ-ary functions F_i of (5.5),(5.6) of [ShSt1]. This reformulation will be implicit in what follows, so we do not enter into the details of the reformulation.

(1.6) The pair $(\mathcal{F}, X)$ is suitable if card $X = \lambda, \lambda+1 \cup \{\mathbb{P}\} \subseteq X$ and X is the underlying set of an $L_{\lambda\lambda}$-elementary substructure of a sufficiently large model (so that, among other things, X is closed for sequences of length $< \lambda$), and $\mathcal{F}$ is an X-normal filter on $\mathcal{P}(\lambda) \cap X$.

Definition: $\mathbb{P}$ is $(\mathcal{F}, X)$-c.c. iff whenever $S, \vec{p} \in X$ are as in (1.3), there's $\mathcal{F}$-positive $S^1 \in X$ and $\bar{p} \in P \cap X$ such that $\vec{p} \mid S^1$ is $\bar{p}$-orderly, in the sense of X (i.e. $\bar{p} \leq p_\alpha$, for all $\alpha \in S^1$, is the largest condition not only in X but in all of $\mathbb{P}$ with this property and for all $q \in P \cap X$, there's $C \in \mathcal{F}$ (and therefore in X) such that q is compatible with $\bar{p}$ iff for all $\alpha \in C \cap S^1$, q is compatible with p_α). ∎

(1.7) Definition: $\mathbb{P}$ is λ-tame iff for all suitable $(\mathcal{F}, X)$, $\mathbb{P}$ is $(\mathcal{F}, X)$-c.c. ∎

(1.8) We now have the preservation theorem for λ-tameness which is in complete analogy with the Theorem of (1.4).

THEOREM: *If $Q(\lambda, \alpha)$, where $\alpha < \lambda$, $\mathbb{P}$ is λ-tame and $\alpha + 1$-strategically-closed, then in $V^{\mathbb{P}}$, $Q(\lambda, \alpha)$.*

Proof: Given the name, $\mathfrak{M}$ of a large model, and the name $\mathring{Y}$ of the right kind of elementary substructure, we seek a name, $\mathring{X}$, for a larger, appropriately elementary substructure of some (named) large model, for which we can find the name, $\mathring{\mathcal{F}}$, of a filter, such that in $V^{\mathbb{P}}$, $(\mathring{\mathcal{F}}, \mathring{X})$ is suitable and NONEMPTY has a winning strategy in $G^{\mathring{X}}(\lambda, \mathring{\mathcal{F}}, \alpha)$.

So, let $\mu > 2^{2^{2^{\max(\lambda, \mathrm{card}(P))}}}$, with $\mathbb{P}, \mathfrak{M}, \mathring{Y} \in H_\mu$ (as names) and let N be the underlying set of a size λ $L_{\lambda\lambda}$-elementary substructure of $(H_\mu, \in)$, with $\mathbb{P}, \mathfrak{M}, \mathring{Y} \in N$. Let $\mathcal{F}$ be obtained for this N as was E in (5.6) of [ShSt1]. We shall take $\mathring{X}$ to be $N[\mathring{G}]$ and $\mathring{\mathcal{F}}$, as always, to be the canonical name for the filter generated by $\mathcal{F}$ in $N[\mathring{G}]$. From here on, we mimic the proof of (1.4), noting that it can be carried out over N.

(1.9) We now present, with his permission, Levinski's result which shows that we must assume some form of pseudo-closure property of $\mathbb{P}$ in order to prove the main preservation result, (1.4). We also present, but without proof, some other results communicated by Levinski on related questions.

LEMMA (Levinski): *Let $\theta \leq \rho < \lambda$. If there is a filter, $\mathcal{F}$, on λ which is λ-complete, non-trivial and for which $\mathbb{P}(\mathcal{F})$ is $(\leq \theta, \infty)$-distributive, then $\rho^\theta < \lambda$*

Proof: Fix an $\mathcal{F}$ as in the hypotheses and towards a contradiction, suppose the conclusion fails. Let $(a_\alpha : \alpha < \lambda)$ be an enumeration without repetitions of a subset of $[\rho]^\theta$. Of course such an $\mathcal{F}$ is precipitous, so let D be generic for the partial ordering of $\mathcal{F}$-positive sets with reverse inclusion and, in $V[D]$, let N be the transitive $\in$-model isomorphic to the generic ultrapower, $(V^{\lambda} \cap V)/D$, and let j be the generic embedding of V into N, so j is $h \circ i$ where h is the transitivizing isomorphism from the well-founded ultrapower and i is the canonical embedding into the ultrapower. Let $S = h([a_\alpha : \alpha < \lambda)]_D)$. Since $j|\lambda = \mathrm{id}|\lambda$ (by λ-completeness, as usual), in N, $S \in [\rho]^\theta$. But then $S \in V[D] \cap [\rho]^\theta$, so by the hypothesis of $(\leq \theta, \infty)$-distributivity of $\mathbb{P}(\mathcal{F})$, $S \in V$. WLOG, we can assume that D was V-normal, so that $h([\mathrm{id}|\lambda]_D) = \lambda$; if this failed, redo the argument replacing D by the V-ultrafilter (which lies in $V[D]$, and conversely) obtained by reading off membership in the ultrafilter according to whether or not the image under j contains λ. Then, as usual, we have that $S = (j(a_\alpha : \alpha < \lambda))_\lambda$. Let $x = \{\alpha < \lambda : S = a_\alpha\}$, so of course $x \in D$, which is impossible, since D is non-trivial and $(a_\alpha : \alpha < \lambda)$ was an enumeration without repetitions.

Similar arguments show:

(1) If ρ is regular, $\lambda = \rho^+$ and $\mathcal{F}$ is a λ-complete filter on λ then $\mathcal{F}$ is not $\rho+1$-strategically-closed.

(2) Let λ be measurable, $\rho < \lambda$, ρ regular. Let U be a normal ultrafilter on λ, and let $\mathbb{P}$ be either the Levy collapse of λ to be ρ^+, or the partial ordering for adding μ Cohen subsets of ρ where $\lambda \leq \mu$. In either case, let $\mathcal{F}$ be the filter generated in V^P by U and let $\mathbb{Q}$ be $\mathbb{P}(\mathcal{F})$. Then $\mathbb{Q}$ has a ρ-closed (i.e. $< \rho$-closed) dense subset but is not $(\leq \rho, \infty)$-distributive.

In fact, Levinski has shown that in the Lemma and in (1) above, the λ-completeness of the filter $\mathcal{F}$ can be weakened to the ρ^+-completeness of $\mathcal{F}$, using a somewhat more careful analysis.

2. Lifting positive relations to singular cardinals with a system of strong filters

Throughout this section, the hypotheses (1)–(4), below, are in effect. See the Introduction for a discussion of these hypotheses in the context of Theorem 6 of [ShSt1]. Note that in (2.3) below, the runs of the games necessary to obtain the a_α of the (CP), below, MUST have length $\theta + 1$, although for the $b_\alpha \subseteq a_\alpha$ it would suffice to have these only for α in a subset of θ of size $\aleph_1$, for which games of length $\aleph_1$ would suffice.

(1) θ is regular, $\theta > \mu$, $\theta \to (\theta, \delta)^2$ and $\theta > 2^{<\delta}$.

(2) $\mu_0 = 0$; $(\lambda_\alpha : \alpha < \theta)$ is monotone increasing with each λ_α regular, $\lambda_0 > \theta$, $\mu_\alpha = \sup\{\lambda_\beta : \beta < \alpha\}$ (so $\mu_\alpha < \lambda_\alpha$).

(3) Each $\lambda_\alpha \to (\lambda_\alpha, \delta)^2$ (in our context, this will follow, since we'll have $\delta =$ the base λ_α logarithm of λ_α, viz. [EHMR, section 17].

(4) $\lambda = \sup\{\lambda_\alpha : \alpha < \theta\}$.

Our work will center around the following Canonization Property.

(CP): Whenever $F : [\lambda]^2 \to \theta$, there's $f : [\theta]^2 \to \theta$, such that

(i) there are $a_\alpha \subseteq [\mu_\alpha, \lambda_\alpha)$, card $a_\alpha = \lambda_\alpha$, for $\alpha < \theta$, such that if $\alpha < \beta < \theta$ and $f(\alpha, \beta) = 0$, then for all $(x, y) \in a_\alpha \times a_\beta$, $F(x, y) = 0$,

(ii) there are $b_\alpha \subseteq a_\alpha$, card $b_\alpha = \theta$ and if $\alpha < \beta < \theta$ then for all $(x, y) \in b_\alpha \times b_\beta$, $F(x, y) = f(\alpha, \beta)$ (*Remark*: for our intended applications it will suffice to have b_α non-empty).

(2.1) LEMMA: (CP)$\Rightarrow \lambda \to (\lambda, \delta)^2$.

Proof: We shall apply the (CP) with F: $[\lambda]^2 \to 2$, obtaining $f : [\theta]^2 \to 2$, (a_α, b_α), $\alpha < \theta$, as in (CP). Note that if for some $\alpha < \theta$, there's $a'_\alpha \in [a_\alpha]^\delta$ s.t. $F''[a'_\alpha]^2 = \{1\}$, then a'_α is sufficiently large homogeneous for F and color 1, so suppose, WLOG, there is no such a'_α. Then, since $\lambda_\alpha \to (\lambda_\alpha, \delta)^2$, there is $a'_\alpha \in [a_\alpha]^{\lambda_\alpha}$ s.t.
$F''[a'_\alpha]^2 = 0$.

Since $\theta \to (\theta, \delta)^2$, suppose, first, that there's $X \in [\theta]^\theta$, s.t. $f''[X]^2 = \{0\}$; then, for $\alpha < \beta$, both in X, $F''a_\alpha \times a_\beta = \{0\}$. But then, letting

$$Y = \bigcup\{a'_\alpha : \alpha \in X\},$$

Y is of size λ and is homogeneous for F and color 0. Thus, we may suppose that there's $X \in [\theta]^\delta$ s.t. $f''[X]^2 = \{1\}$. For $\alpha \in X$, choose $x_\alpha \in b_\alpha$ and let $Y = \{x_\alpha : \alpha \in X\}$. Then, Y is of size δ and is homogeneous for F and color 1. ∎

The remainder of this section will be devoted to showing how to obtain the (CP) from a system of strong filters on the λ_α. Henceforth, assume the filter system hypothesis:

(FSH) for $\alpha < \theta$, $\mathcal{F}_\alpha$ is a λ_α-complete filter on λ_α for which $(\mathcal{F}^+_\alpha, \supseteq)$ is $\theta + 1$-strategically-closed.

Suppose $F : [\lambda]^2 \to \theta$ and $f : [\theta]^2 \to \theta$. The following definitions depend on F, f, but we suppress mention of them.

(2.2) Definition: If $\alpha < \beta < \theta$, $x \in [\mu_\alpha, \lambda_\alpha)$, $d \subseteq [\mu_\beta, \lambda_\beta)$ then

$$d^X =_{\mathrm{def}} \{y \in d : F(x,y) = f(\alpha,\beta)\}.$$

If we start with $d \in \mathcal{F}_\beta^+$, then x is *d-good* if $d^x \in \mathcal{F}_\beta^+$. If $\vec{d}$ is a sequence with domain a (nonempty) subset of (α, θ), then x is *$\vec{d}$-good* if for all $\beta \in \operatorname{dom} \vec{d}$, x is d_β-good. ∎

(2.3) LEMMA: (FSH)$\Rightarrow$ (CP).

Proof: We shall obtain the $a_\alpha \in \mathcal{P}([\mu_\alpha, \lambda_\alpha)) \cap \mathcal{F}_\alpha^+$ with the following additional property: for all $l < \theta$ and all $\alpha < \beta < \theta$:

$(*)_{\alpha\beta l}$ *IF*$(\exists A_\alpha \in (\mathcal{P}(a_\alpha) \cap \mathcal{F}_\alpha^+))(\exists A_\beta \in (\mathcal{P}(a_\beta) \cap \mathcal{F}_\beta^+))l \notin F''A_\alpha \times A_\beta$, *THEN* $l \notin F''a_\alpha \times a_\beta$ (*Remark:* This is similar to the property (*) of [ShSt1, (5.2)]).

Assuming we have the a_α with these additional properties, we complete the proof of (CP); we will then construct the a_α much as in [ShSt1, (5.2)], using runs of the games $G(\lambda_\alpha, \mathcal{F}_\alpha, \theta + 1)$. We first define $f(\alpha, \beta) = 0$ iff $F''a_\alpha \times a_\beta = \{0\}$; otherwise, $f(\alpha,\beta) = \min(F''a_\alpha \times a_\beta \setminus \{0\})$.

Thus, if $f(\alpha, \beta) = l > 0$, then, whenever $a \in \mathcal{P}(a_\alpha) \cap \mathcal{F}_\alpha^+$ and $d \in \mathbb{P}(a_\beta) \cap F_\beta^+$, $l \in F''a \times d$. Thus, (i) of the (CP) is clear. Before defining the b_α we need a

PROPOSITION: *If $\alpha < \beta < \theta$, $a \in \mathcal{P}(a_\alpha) \cap \mathcal{F}_\alpha^+$, $d \in \mathcal{P}(a_\beta) \cap \mathcal{F}_\beta^+$ then $\{x \in a : x$ is d-good$\} \in \mathcal{F}_\alpha^+$.*

Proof: Suppose not. Then, $c =_{\mathrm{def}} \{x \in a : d^x = \emptyset (\mathrm{mod}\ \mathcal{F}_\beta)\}$ is $\mathcal{F}_\alpha$-almost all of a. In particular, c is $\mathcal{F}_\alpha$-positive. Note that $c' =_{\mathrm{def}} \bigcup\{d^x : x \in c\} = \emptyset (\mathrm{mod}\ \mathcal{F}_\beta)$, since $\mathcal{F}_\beta$ is λ_β-complete. But we now get a contradiction to $(*)_{\alpha\beta f(\alpha,\beta)}$, with $A_\alpha = c$, $A_\beta = d \setminus c'$, since then, if $(x,y) \in A_\alpha \times A_\beta$, we have $x \in c$, $y \notin d^x$, so $F(x,y) \neq f(\alpha,\beta)$. ∎

COLLORARY: *If $\alpha < \beta < \theta$, $a \in \mathcal{P}(a_\alpha) \cap \mathcal{F}_\alpha^+$, $\vec{d}$ is a sequence with domain a (nonempty) subset of (α, θ) such that for $\beta \in \operatorname{dom} \vec{d}$, $d_\beta \in \mathbb{P}(a_\beta) \cap \mathcal{F}_\beta^+$, then $\{x \in a : x$ is $\vec{d}$-good $\} \in \mathcal{F}_\alpha^+$.*

Proof: If not, then $c =_{\mathrm{def}} \{x \in a : x$ is not $\vec{d}$- good$\}$ is $\mathcal{F}$-almost all of a and it is the union over $\beta \in \operatorname{dom} \vec{d}$ of the $\{x \in a : x$ is not d_β-good$\}$. But dom $\vec{d}$ has power $\leq \theta < \lambda_\alpha$ and $\mathcal{F}_\alpha$ is λ_α-complete, so some $\{x \in a : x$ is not d_β -good$\}$ is $\mathcal{F}$-positive. Now proceed as in the proof of the Proposition. ∎

We're now ready to define the b_α. We define recursively for $\alpha \le \beta < \theta$, $a_{\alpha\beta} \in \mathcal{P}(a_\beta) \cap \mathcal{F}^+_\beta$, $b_\alpha \in [a_\alpha]^\theta$; for $\alpha = 0$, we set $\bar{a}_{0\beta} = a_\beta$; if $0 < \alpha \le \beta < \theta$, we set $\bar{a}_{\alpha\beta} = \bigcap\{a_{\gamma\beta} : \gamma < \alpha\}$. We shall have:

(i) $b_\alpha \subseteq \bar{a}_{\alpha\alpha}$,

(ii) for all $0 < \beta < \theta$, $(a_{\alpha\beta} : \alpha \le \beta)$ are NONEMPTY's plays according to her winning strategy in a run of $G(\lambda_\beta, \mathcal{F}_\beta, 2(\beta+1))$ starting from a_β; thus this sequence is $\subseteq$-decreasing,

(iii) for $\alpha < \beta < \theta$, $(\forall (x,y) \in b_\alpha \times a_{\alpha\beta}) F(x,y) = f(\alpha,\beta)$; NOTE THAT b_α satisfies this FOR ALL $\alpha < \beta < \theta$.

To accomplish this, we shall define by recursion on $\xi < \theta$ $(x^\xi_\alpha : \xi < \theta)$, $(a^\xi_\alpha : \xi < \theta)$ and for $\alpha < \beta < \theta$, $(a^\xi_{\alpha\beta} : \xi < \theta)$. We set $\bar{a}^0_\alpha = \bar{a}_{\alpha\alpha}$, $\bar{a}^0_{\alpha\beta} = \bar{a}_{\alpha\beta}$, and for $0 < \xi < \theta$, we set $\bar{a}^\xi_\alpha = \bigcap\{a^\zeta_\alpha : \zeta < \xi\}$, $\bar{a}^\xi_{\alpha\beta} = \bigcap\{a^\zeta_{\alpha\beta} : \zeta < \xi\}$. We shall have the following properties:

(a) $x^\xi_\alpha \in \bar{a}^\xi_\alpha \in \mathcal{F}^+_\alpha$,

(b) the $(a^\xi_\alpha : \xi < \theta)$, $(a^\xi_{\alpha,\beta} : \xi < \theta)$ are NONEMPTY's plays according to her winning strategy in runs of the $G(\lambda_\alpha, \mathcal{F}_\alpha, \theta+1)$, respectively the $G(\lambda_\beta, \mathcal{F}_\beta, \theta+1)$, starting from the $\bar{a}^0_\alpha = \bar{a}_{\alpha\alpha}$, respectively the $\bar{a}^0_{\alpha\beta}$; in particular they form $\subseteq$-decreasing sequences of $\mathcal{F}_\alpha$-positive, respectively $\mathcal{F}_\beta$ -positive, sets.

Thus, it remains only to describe EMPTY's plays. At stage ξ, EMPTY chooses $x^\xi_\alpha \in \bar{a}^\xi_\alpha$ which is $\vec{d}^\xi$-good, where $\vec{d}^\xi =_{\rm def} (\bar{a}^\xi_{\alpha\beta} : \alpha < \beta < \theta)$. EMPTY then plays the $e^\xi_{\alpha\beta}$ $(\alpha < \beta)$, which are, by definition, $\{y \in \bar{a}^\xi_{\alpha\beta} : F(x^\xi_\alpha, y) = f(\alpha,\beta)\}$. Finally, EMPTY plays the e^ξ_α, the set of $x \in \bar{a}^\xi_\alpha \setminus \{x^\zeta_\alpha : \zeta < \xi\}$ which are $(e^\xi_{\alpha\beta} : \alpha < \beta < \theta)$-good.

By the Corollary to the Proposition, these runs can be carried out. Finally, $b_\alpha = \{x^\xi_\alpha : \xi < \theta\}$ and $a_{\alpha\beta} = \bigcap\{a^\xi_{\alpha\beta} : \xi < \theta\}$. Clearly these are as required. This completes the proof of the (CP). ∎

Finally, we show how to obtain the a_α. Enumerate $\theta \times (< \bigcap \theta^2)$ as $((l_i, \alpha_i, \beta_i) : i < \theta)$, with θ repetitions of each triple. Let $\bar{a}^0_\alpha = [\mu_\alpha, \lambda_\alpha)$. For $0 < i < \theta$, let $\bar{a}^i_\alpha = \bigcap\{a^j_\alpha : j < i\}$. Then, $(a^i_\alpha : i < \theta)$ is the sequence of NONEMPTY's plays by her winning strategy in a run of $G(\lambda_\alpha, \mathcal{F}_\alpha, \theta+1)$. In particular this is a $\subseteq$-decreasing sequence of $\mathcal{F}_\alpha$-positive sets. Thus, we need only describe EMPTY's plays. At stage i, if $(\alpha,\beta) \ne (\alpha_i, \beta_i)$ then EMPTY repeats the previous move. Otherwise, EMPTY checks to see whether there's $A \subseteq \bar{a}^i_\alpha$, $B \subseteq \bar{a}^i_\beta$ such that $l_i \notin F''A \times B$. If so, EMPTY plays $e^i_\alpha = A$, $e^i_\beta = B$, for some such pair (A,B);

if not, he repeats the previous move. This complete the proof of the Theorem and as outlined in the Introduction, the proof of Theorem 6 of [ShSt1]. ∎

3. An elementary proof

In this section we prove some positive ordinal relations related to the material of section 2 of [ShSt1]: since we cannot always prove, e.g., $\omega_\beta\omega_\alpha \to (\omega_\beta\omega_\alpha, 3)^2$, we thought to ask how large the second cardinal on the left side of the arrow must be made, while keeping the first cardinal and the right side fixed, in order to get a provable positive relation. Our results, under GCH (but this is really just for notational simplicity), are best possible (recall that the second factor on the left side is required to be a **cardinal**) in view of section 2 of [ShSt1]. In this section, when defining colorings by two colors we follow our usual convention that the first color is red and the second is green.

(3.1) For notational simplicity, assume GCH (we shall indicate below where this is used; at this point it will be clear how to eliminate it).

THEOREM: *Let $2 \le k < \omega$. If $\aleph_\beta$ is regular and $\beta > \alpha + k - 2$, then:*

$$\omega_\beta\omega_{\alpha+k-2} \to (\omega_\beta\omega_\alpha, k)^2.$$

Proof: We work by induction on k, the case $k = 2$ being trivial, so let $k > 2$, and suppose that the induction hypothesis holds for $k-1$, i.e., that $\omega_\beta\omega_{\alpha+k-3} \to (\omega_\beta\omega_\alpha, k-1)^2$. We view $\omega_\beta\omega_{\alpha+k-2}$ as $A =_{\text{def}} \omega_\beta \times \omega_{\alpha+k-2}$. Let $c : [A]^2 \to \{\text{red}, \text{green}\}$ be given. By some preliminary applications of the Erdös–Dushnik–Miller theorem which will be quite familiar to the reader of [ShSt1], we may assume that for all $i < \omega_{\alpha+k-2}$ and all $\xi < \zeta < \omega_\beta$, $c\{(\xi,i),(\zeta,i)\}$ =red. For $(\zeta, i) \in A$, let $S_{(\zeta,i)} = \{j \in \omega_{\alpha+k-2}\setminus\{i\} : (\exists^{\aleph_\beta}\xi)(c\{(\zeta,i),(\xi,j)\} = \text{green}\}$. We consider two cases.

CASE 1: For some (ζ, i), $S_{(\zeta,i)}$ has power $\aleph_{\alpha+k-3}$.

In case 1, fix such a (ζ, i) and for $j \in S_{(\zeta,i)}$, let $Y_j \in [\omega_\beta]^{\aleph_\beta}$ be such that for all $\xi \in Y_j$, $c\{(\zeta,j),(\xi,j)\}$ =green, and apply the induction hypothesis to the coloring $c|[\bigcup\{Y_j \times \{j\} : j \in S_{(\zeta,i)}\}]^2$; a homogeneous red $\omega_\beta\omega_\alpha$ is as required, while if a is a green $k-1$ set, then $a \cup \{(\zeta, i)\}$ is a green k-set for c, again as required.

CASE 2: There is no such (ζ, i).

In case 2, each $S_{(\zeta,i)}$ has power $< \aleph_{\alpha+k-3}$ and since there are at most $\aleph_{\alpha+k-2}$ subsets of $\omega_{\alpha+k-2}$ of power $< \aleph_{\alpha+k-3}$ (here is one application of GCH, in fact far less; there is another below), for each i, there are S_i, and $Y_i \in [\omega_\beta]^{\aleph_\beta}$ such that for $\zeta \in Y_i$, $S_{(\zeta,i)} = S_i$. By Hajnal's free subset result, section 44 of [EHMR], there is $T \in [\omega_{\alpha+k-2}]^{\aleph_{\alpha+k-2}}$ such that for $i \neq j$, both from $T, i \notin S_j$ (here is the other application of GCH). Now, by recursion on $(l,i) \in \omega_\beta \times T$, with respect to the lexicographic ordering, we define $\zeta_{l,i} \in Y_i$, such that for fixed i the $\zeta_{l,i}$ are monotone increasing. We shall have that $\bigcup\{\{(\zeta_{l,i},i) : l < \omega_\beta\} : i \in T\}$ will be a homogeneous red $\omega_\beta\omega_{\alpha+k-2}$, which is more than required. Note that, for fixed $i \in T$ and $\zeta \in Y_j$, for $j \in T\backslash\{i\}$, there is $\theta = \theta_{(\zeta,i),j} < \omega_\beta$ such that for $\xi \in Y_j\backslash\theta$, $c\{(\zeta,i),(\xi,j)\}$ =red. The definition of the $\zeta_{l,i}$ is now clear: we choose $\zeta_{l,i} \in Y_i$ greater than all the $\zeta_{m,j}$ and all the $\theta_{(\zeta_{m,j},j),i}$, for (m,j) lexicographically less than (l,i). Clearly this suffices and completes the proof of the Theorem in case 2 and thus the proof of the Theorem. ∎

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